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Applicability of Weibull analysis for brittle materials

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Weibull analysis for the interpretation of strengths of brittle materials was previously justified for a particular flaw size distribution. The present results show that the Weibull distribution provides a close approximation to the distribution of failure stress for all the flaw size distributions considered. However certain reservations are noted in the interpretation of the Weibull modulus.

1. Introduction
The Weibull distribution is commonly used in the analysis of strengths of brittle materials. It has provided a means of comparing the strengths of different brittle materials and is important in the interpretation of the strength of a material and in the design of components for engineering applications.

Jayatilaka and Trustrum [1] showed that for a particular flaw size distribution, the distribution of failure stress under uniaxial tensile loading is closely approximated by the Weibull distribution, provided that the number of flaws or cracks in the material is sufficiently large. It is possible that the flaw size distribution of a material may have a shape other than previously described [1]. For example the distribution could take a normal or exponential shape. It is well known that the same material, prepared under different conditions, has different values of the Weibull modulus, m. In this paper other possible flaw size distributions are considered and the corresponding distributions of failure stress are derived. Thereby it is possible to comment on the applicability of Weibull analysis for the strength of brittle materials.

2. Theory
The strength, σ, of a crack of size c inclined at an angle β to a uniform tensile stress may be approximately expressed for a material with Poisson's ratio 0.25 [1] as

$$\sigma^2 c = K^2 \beta^{-1} \tag{1}$$

where K is the critical stress intensity factor of the material (which is the same as K_{IC}). Let $H(c)$ be the cumulative distribution function of the crack size then, assuming all crack angles are equally likely, the probability of failure for one crack at stress σ, $F(\sigma)$, is given by

$$\begin{aligned} F(\sigma) &= \int_0^{\pi/2}\int_x^{\infty} \frac{2}{\pi} H'(c)\,\mathrm{d}c\,\mathrm{d}\beta \\ &= \int_0^{\pi/2} \frac{2}{\pi}[1 - H(x)]\,\mathrm{d}\beta \end{aligned} \tag{2}$$

where $x = K^2/\beta\sigma^2$ and $H'(c)$ is the probability density function. Making the further assumption that fracture at the flaw with minimum strength leads to total failure, it follows that the distribution function of the failure stress for N cracks, $G(\sigma)$, satisfies

$$\begin{aligned} G(\sigma) &= 1 - \text{probability all } N \text{ cracks survive at stress } \sigma \\ &= 1 - [1 - F(\sigma)]^N. \end{aligned} \tag{3}$$

De Haan [2] derives the limit distributions for the maxima of identically, independently distri-

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buted random variables and gives sufficient conditions for the convergence of the maximum to a specified limit distribution. His work is based on earlier work by Gnedenko and Von Mises. Since the minimum flaw strength is required a slight adaptation of the results in De Haan's paper is necessary. This leads to sufficient conditions for the convergence of the minimum of a set of identically, independently distributed random variables, namely the flaw strengths, to a given limit distribution and the following theorem contains the required results.

2.1. Theorem

(i) Suppose $F'(\sigma) > 0$ in (σ_u, σ_v), $F'(\sigma) = 0$ for $\sigma < \sigma_u$ and for some $\alpha > 0$

$$\lim_{\sigma \downarrow \sigma_u} (\sigma - \sigma_u) \frac{F'(\sigma)}{F(\sigma)} = \alpha$$

then for large N

$$G(\sigma) \sim 1 - \exp\left[-\left(\frac{\sigma - \sigma_u}{a_N}\right)^{\alpha}\right] \quad (\sigma > \sigma_u)$$

where $F(\sigma_u + a_N) = N^{-1}$.

(ii) Suppose $F''(\sigma) > 0$ in (σ_u, σ_v), $F'(\sigma) = 0$ for $\sigma \leqslant \sigma_u$ and

$$\lim_{\sigma \downarrow \sigma_u} \frac{\phi''(\sigma)}{[\phi'(\sigma)]^2} = 0, \quad \text{where} \quad \phi(\sigma) = \ln F(\sigma),$$

then for large N

$$G(\sigma) \sim 1 - \exp\left[-\exp\left(\frac{\sigma - b_N}{a_N}\right)\right] \quad (\sigma_u < \sigma < \infty),$$

where $\phi(b_N) = -\ln N$ and $a_N = 1/\phi'(b_N)$.

The limit distribution in part (i) of the theorem is the Weibull distribution, where α is the Weibull modulus, σ_u is the stress below which there is a zero probability of failure, N is the number of cracks and the probability of failure of one crack at stress $\sigma_u + a_N$ is $1/N$. In part (ii) the limit distribution is known as the Gumbel distribution which has mean $b_N - a_N\gamma$ and variance $\pi^2 a_N^2/6$, where $\gamma = 0.5772$ is Euler's constant, so a_N is a scale parameter and the probability of failure of one crack at stress b_N is $1/N$.

3. Application

Four different types of flaw size distribution are now considered and the corresponding limit distributions for the failure stress are derived. Fig. 1 shows examples of the four distribution types, which all have the same mean and variance for ease of comparison.

3.1. Case a $\quad 1 - H(c) \sim kc^{-n}$ $(n > 0)$ for large c

This case includes the Pareto, Cauchy, t and F distributions and also the distribution used by Jayatilaka and Trustrum [1]. Substituting $H(c)$ into Equation 2 gives

$$F(\sigma) \sim \int_0^{\pi/2} \frac{2}{\pi} k \left(\frac{K^2}{\beta\sigma^2}\right)^{-n} d\beta = k'\sigma^{2n} \text{ as } \sigma \downarrow 0 \quad (4)$$

where $k' = [k/(n+1)](\pi/2K^2)^n$. On applying theorem (i) with $\sigma_u = 0$ gives

$$\lim_{\sigma \downarrow 0} \frac{\sigma F'(\sigma)}{F(\sigma)} = 2n > 0 \quad \text{and} \quad k' a_N^{2n} \sim N^{-1}$$

hence

$$G(\sigma) \sim 1 - \exp(-k'N\sigma^{2n}) \qquad \text{for large } N. \quad (5)$$

So for flaw size distributions which decay like an inverse power law, the resulting limit distribution for the failure stress is the Weibull distribution with Weibull modulus $m = 2n$. The mean failure stress $\bar{\sigma}$ is given by

$$\bar{\sigma} = (k'N)^{-1/m}\,\Gamma\left(1 + \frac{1}{m}\right) \quad (6)$$

where Γ is the gamma function. So for two specimens of the same material with N_1 and N_2 flaws, respectively,

$$\frac{\bar{\sigma}_1}{\bar{\sigma}_2} = \left(\frac{N_2}{N_1}\right)^{1/m}. \quad (7)$$

3.2. Case b

$1 - H(c) \sim k(\ln c)^r e^{-\lambda(\ln c)^2}$ $(\lambda > 0)$ for large c

This case covers the Lognormal distribution, which is often used for data with a skew distribution, and it can be shown that

$$F(\sigma) \sim \frac{k}{2\lambda}\left(\ln \frac{2K^2}{\pi\sigma^2}\right)^{r-1} \exp\left[-\lambda\left(\ln \frac{2K^2}{\pi\sigma^2}\right)^2\right] \quad \text{as } \sigma \downarrow 0. \quad (8)$$

An application of part (ii) of the theorem with $\phi(\sigma) \sim -4\lambda(\ln \sigma)^2$ as $\sigma \downarrow 0$ gives

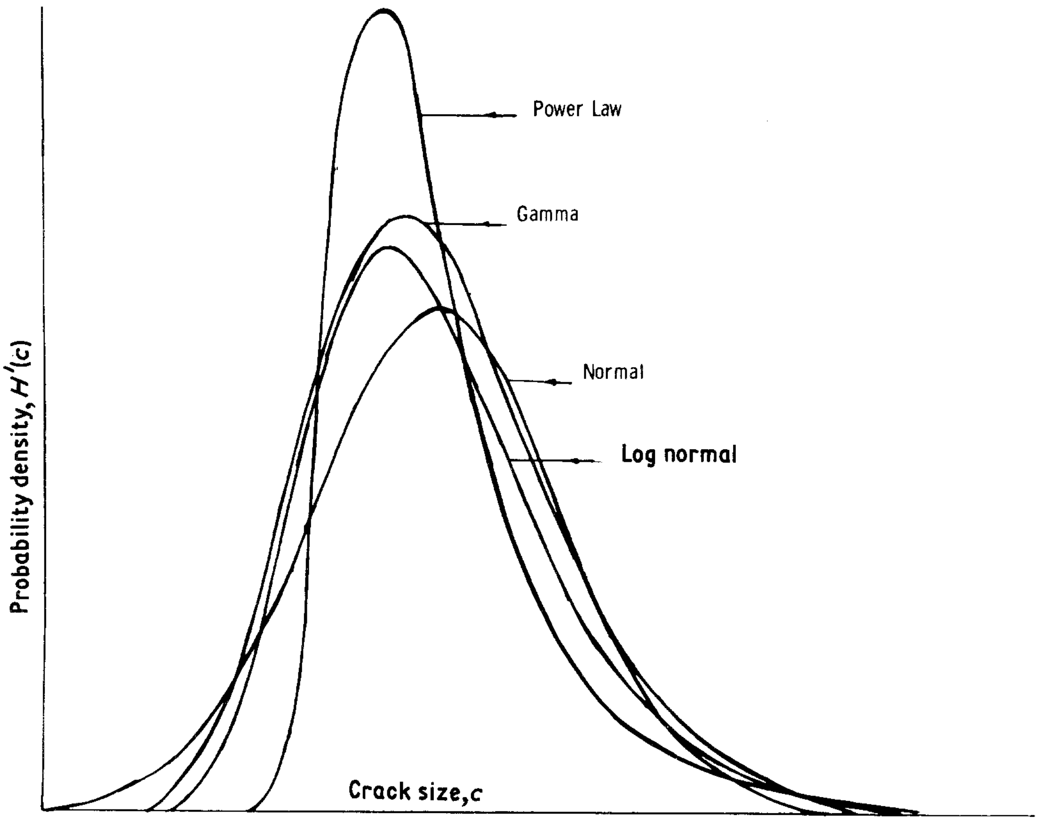

Figure 1 Graphs of Power Law, Lognormal, Gamma and Normal probability densities with equal means and equal variances.

$$\lim_{\sigma\downarrow 0} \frac{\phi''(\sigma)}{[\phi'(\sigma)]^2} = \lim_{\sigma\downarrow 0} \frac{1}{8\lambda}\left[\frac{\ln\sigma - 1}{(\ln\sigma)^2}\right] = 0$$

so

$$G(\sigma) \sim 1 - \exp\left[-\exp\left(\frac{\sigma - b_N}{a_N}\right)\right] \quad \text{for large } N \tag{9}$$

where $4\lambda(\ln b_N)^2 \sim \ln N$ and $a_N \sim -b_N/8\lambda \ln b_N$. The mean failure stress

$$\bar{\sigma} \sim \exp\left[-(\ln N/4\lambda)^{1/2}\right] \quad \text{for large } N \tag{10}$$

and the ratio of mean failure stresses

$$\frac{\bar{\sigma}_1}{\bar{\sigma}_2} \sim \exp\left[\left(\frac{\ln N_2}{4\lambda}\right)^{1/2} - \left(\frac{\ln N_1}{4\lambda}\right)^{1/2}\right]. \tag{11}$$

3.3. Case c $\quad 1 - H(c) \sim kc^r e^{-\lambda c}$ ($\lambda > 0$) for large c

The Exponential, Gamma and χ^2 distributions are examples of this case for which

$$F(\sigma) \sim \frac{k}{\lambda}\left(\frac{2K^2}{\pi\sigma^2}\right)^{r-1} \exp\left(-\frac{2\lambda K^2}{\pi\sigma^2}\right) \quad \text{as } \sigma \downarrow 0. \tag{12}$$

Thus $\phi(\sigma) \sim -2\lambda K^2/\pi\sigma^2$ and applying theorem (ii) gives

$$\lim_{\sigma\downarrow 0} \frac{\phi''(\sigma)}{[\phi'(\sigma)]^2} = \lim_{\sigma\downarrow 0} -\frac{3\pi\sigma^2}{4\lambda K^2} = 0.$$

Hence

$$G(\sigma) \sim 1 - \exp\left[-\exp\left(\frac{\sigma - b_N}{a_N}\right)\right] \quad \text{for large } N \tag{13}$$

where $2\lambda K^2/\pi b_N^2 \sim \ln N$ and $a_N \sim \pi b_N^3/4\lambda K^2$, so

$$\bar{\sigma} \sim (2\lambda K^2/\pi \ln N)^{1/2} \quad \text{for large } N \tag{14}$$

and

$$\frac{\bar{\sigma}_1}{\bar{\sigma}_2} \sim \left(\frac{\ln N_2}{\ln N_1}\right)^{1/2}. \tag{15}$$

3.4. Case d $\quad 1 - H(c) \sim kc^r e^{-\lambda c^2 - \rho c}$ ($\lambda > 0$) for large c

The normal distribution is included in this case for which

$$F(\sigma) \sim \frac{k}{2\lambda}\left(\frac{2K^2}{\pi\sigma^2}\right)^{r-2} \exp\left[-\lambda\left(\frac{2K^2}{\pi\sigma^2}\right)^2 - \rho\,\frac{2K^2}{\pi\sigma^2}\right] \quad \text{as } \sigma \downarrow 0. \tag{16}$$

This gives $\phi(\sigma) \sim -\lambda(2K^2/\pi\sigma^2)^2$ as $\sigma \downarrow 0$ and

$$\lim_{\sigma\downarrow 0} \frac{\phi''(\sigma)}{[\phi'(\sigma)]^2} = \lim_{\sigma\downarrow 0} -\frac{5\pi^2\sigma^4}{16\lambda K^4} = 0.$$

Hence

$$G(\sigma) \sim 1 - \exp\left[-\exp\left(\frac{\sigma - b_{\mathrm{N}}}{a_{\mathrm{N}}}\right)\right] \quad \text{for large } N \tag{17}$$

where $\lambda(2K^2/\pi b_{\mathrm{N}}^2)^2 \sim \ln N$ and $a_{\mathrm{N}} \sim \pi^2 b_{\mathrm{N}}^5/16\lambda K^4$, so

$$\bar{\sigma} \sim \left(\frac{4\lambda K^4}{\pi^2 \ln N}\right)^{1/4} \quad \text{for large } N \tag{18}$$

and

$$\frac{\bar{\sigma}_1}{\bar{\sigma}_2} \sim \left(\frac{\ln N_2}{\ln N_1}\right)^{1/4}. \tag{19}$$

The above cases suggest that the Weibull distribution is the limit distribution only for flaw size distributions which decay like an inverse power law for large flaws, and that faster rates of decay lead to the Gumbel distribution. One disadvantage of the Gumbel distribution as a model for the distribution of failure stress is that, for sufficiently low values of the probability of failure P_{f}, it predicts negative values for the failure stress. However in Section 4, it is shown that a family of Weibull distributions can be found which closely approximate a given Gumbel distribution except for small values of P_{f} and $1-P_{\mathrm{f}}$. This property was noted by Peto and Lee [3] and justifies the use of the Weibull distribution as a model for P_{f}.

An interesting special case occurs when the cracks are of constant size and randomly oriented. This applies to specimens that are subjected to flexural loading and whose surfaces have been polished in a controlled manner to produce constant size scratches on the surfaces. In this case $H(c) = 1$ for $c > c_{\mathrm{u}}$ and zero otherwise, which on substituting into Equation 2 gives $F(\sigma) = 1 - \sigma_{\mathrm{u}}^2/\sigma^2$ for $\sigma > \sigma_{\mathrm{u}} = (2K^2/\pi c_{\mathrm{u}})^{1/2}$ and zero otherwise. The limiting distribution is the Exponential distribution given by

$$G(\sigma) = 1 - \left(\frac{\sigma_{\mathrm{u}}}{\sigma}\right)^{2N} \sim 1 - \exp\left[-2N\left(\frac{\sigma - \sigma_{\mathrm{u}}}{\sigma_{\mathrm{u}}}\right)\right]$$
$$(\sigma > \sigma_{\mathrm{u}} > 0)$$

which is a special case of the Weibull distribution with $m = 1$. In general, Weibull distributions with $\sigma_{\mathrm{u}} > 0$ derive from crack size distributions with an upper bound on the crack size.

4. Relationship between Weibull and Gumbel distributions

Consider the usual form of the Weibull distributions used for determining the strength σ of a brittle material of volume V,

$$P_{\mathrm{f}} = 1 - \exp\left[-V\left(\frac{\sigma - \sigma_{\mathrm{u}}}{\sigma_0}\right)^m\right] \quad (\sigma > \sigma_{\mathrm{u}}) \tag{20}$$

which has mean

$$\bar{\sigma} = \sigma_{\mathrm{u}} + \sigma_0 V^{-1/m}\Gamma(1 + 1/m) \tag{21}$$

and variance

$$\mathrm{Var}(\sigma) = \sigma_0^2 V^{-2/m}[\Gamma(1 + 2/m) - \Gamma^2(1 + 1/m)]. \tag{22}$$

Using the approximation $\exp(x) \triangleq 1 + x$ for $|x| \ll 1$,

$$V^{1/m}\left(\frac{\sigma - \sigma_{\mathrm{u}}}{\sigma_0}\right) = 1 + \frac{\sigma - \sigma_{\mathrm{u}} - \sigma_0 V^{-1/m}}{\sigma_0 V^{-1/m}}$$
$$\triangleq \exp\left[\frac{\sigma - \sigma_{\mathrm{u}} - \sigma_0 V^{-1/m}}{\sigma_0 V^{-1/m}}\right]$$

for $\sigma_{\mathrm{u}} \ll \sigma \ll \sigma_{\mathrm{u}} + 2\sigma_0 V^{-1/m}$, and substituting in Equation 20 gives

$$P_{\mathrm{f}} \triangleq 1 - \exp\left[-\exp m\left(\frac{\sigma - \sigma_{\mathrm{u}} - \sigma_0 V^{-1/m}}{\sigma_0 V^{-1/m}}\right)\right]. \tag{23}$$

The right hand side of Equation 23 is the Gumbel distribution, which has mean

$$\bar{\sigma} = \sigma_{\mathrm{u}} + \sigma_0 V^{-1/m}(1 - \gamma/m) \quad (\gamma = 0.5772) \tag{24}$$

and variance

$$\mathrm{Var}(\sigma) = \pi^2\sigma_0^2 V^{-2/m}/6m^2. \tag{25}$$

Using the results $\Gamma'(1) = -\gamma$ and $\Gamma''(1) = \pi^2/6 + \gamma^2$, one can show that Equation 24 is the first two terms of the Taylor series in m^{-1} of Equation 21 and Equation 25 is the first non-vanishing term of the Taylor series of Equation 22. This means that the approximation is better for larger values of m. Also eliminating V between Equations 24 and 25 gives

$$\bar{\sigma} = \sigma_{\mathrm{u}} + \frac{[6\,\mathrm{Var}(\sigma)]^{1/2}}{\pi}(m - \gamma) \tag{26}$$

which shows that a family of Weibull distributions, with σ_{u} and m linearly related, approximate the Gumbel distribution determined by the values of $\bar{\sigma}$ and $\mathrm{Var}(\sigma)$. This explains a result found by

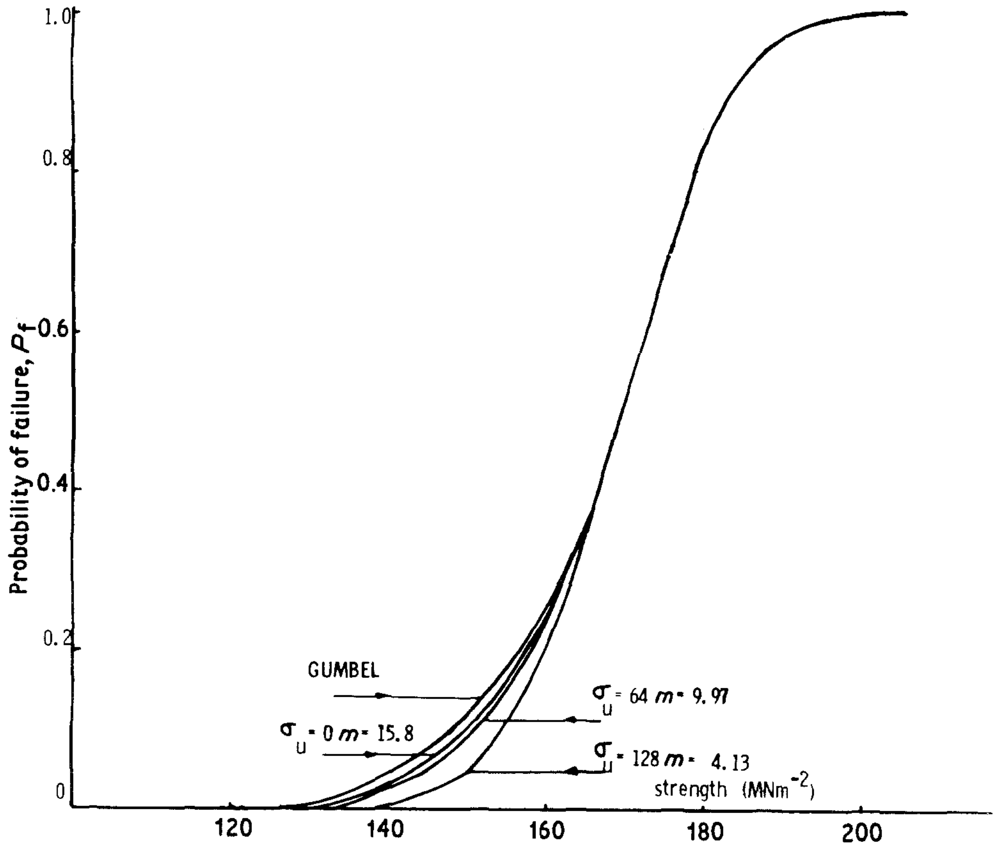

Figure 2 Graphs of the Gumbel and three Weibull distributions with parameters satisfying Equations 24 and 25 with $\bar{\sigma} = 167.0$ and $\text{Var}(\sigma) = 198.2$.

Trustrum and Jayatilaka [4]. It also gives further support for the recommendation in the same paper for using Equation 20 with $\sigma_u = 0$ for the probability of failure at stress σ, since the best approximation to the Gumbel distribution is obtained when m is largest, which occurs at $\sigma_u = 0$. In Fig. 2 three Weibull and one Gumbel distributions are drawn with parameters satisfying Equations 24 and 25, where $\bar{\sigma}$ and $\text{Var}(\sigma)$ are estimated from 32 experimentally observed failure stresses.

5. Safety factors

In engineering designs "safety factors" are commonly used. For brittle materials the safety factor, s, is usually defined as $\bar{\sigma}/\sigma$, where $\bar{\sigma}$ is the mean failure stress and σ is the design stress at a given probability of failure, P_f. The safety factor for the Gumbel distribution can be obtained from Equations 23 and 24 and for $\sigma_u = 0$ is given by

$$s_G = \frac{m - \gamma}{m + \ln[-\ln(1 - P_f)]} \quad (\gamma = 0.5772). \tag{27}$$

For the Weibull distribution with $\sigma_u = 0$, it follows from Equations 20 and 21 that the safety factor is

$$s_W = \frac{\Gamma(1 + 1/m)}{[-\ln(1 - P_f)]^{1/m}} \tag{28}$$

Table I shows the safety factors for the two distributions for a range of values of m and P_f. For very low values of P_f, Equation 27 predicts negative values for s_G, which are unacceptable. The dis-

TABLE I Safety factors for Gumbel distribution and for Weibull distribution in brackets. Negative values are denoted by –

m	P_f		
	$1/10^2$	$1/10^4$	$1/10^6$
6	3.87	–	–
	(2.00)	(4.31)	(9.28)
10	1.75	11.93	–
	(1.51)	(2.39)	(3.79)
12	1.54	4.09	–
	(1.41)	(2.06)	(3.03)
14	1.43	2.80	72.76
	(1.34)	(1.86)	(2.58)

crepancy between the values of s_G and s_W is greater for the lower values of both m and P_f. This is also clear from Fig. 2.

6. Discussion

Experimental studies are not yet sufficiently advanced to determine the crack size distribution in a wide range of brittle materials. It is likely that different materials or the same material prepared in different ways could have widely differing crack size distributions. For example they need not decay like c^{-n} for large c, a shape used in [1] that justified the use of Weibull analysis. The work carried out in this paper examines a range of flaw size distributions and the results support the use of Weibull analysis for all the flaw size distributions considered. Therefore it is possible to conclude that the distribution of failure stress is insensitive to the flaw size distribution, a result which simplifies the analysis of the strengths of brittle materials.

However certain reservations must be noted. When the flaw size distribution $H(c) \sim 1 - kc^{-n}$ as in Case a, n can be related to the Weibull modulus by $m = 2n$. Therefore m is a material parameter determined by the flaw size distribution. In the case of other distributions this is not so and comparing Equation 23 with $\sigma_u = 0$, the Gumbel approximation to the Weibull distribution, and Equation 9 identifies $m = b_N/a_N$. So for Case b, the Lognormal distribution, $m \sim 4\lambda^{1/2}(\ln N)^{1/2}$ and depends on both the number of cracks, N, in the material and a parameter, λ, of the flaw size distribution. In Case c, $m \sim 2 \ln N$ and in Case d, $m \sim 4 \ln N$, so when the flaw size distribution is Exponential, Gamma or Normal, the Weibull modulus is related to $\ln N$ only. Since N is proportional to the volume, m is related to the volume. These results can be used to investigate the nature of the flaw size distribution experimentally.

It is important to note that the ratio of mean strengths of two sets of specimens of the same material with different volumes, given by Equations 7, 11, 15 and 19, is dependent on the type of flaw size distribution. Also the derivation of safety factors based on the Gumbel distribution leads to difficulties as the probability density is nonzero the range $-\infty < \sigma < \infty$. This means that the Gumbel distribution should only be used for $\sigma > 0$ and should be used with care near $\sigma = 0$, where the approximation is poor. For the Gumbel distribution shown in Fig. 2, which was fitted to 32 observed failure stresses, $P_f = 1.4 \times 10^{-7}$ when $\sigma = 0$. This indicates that in practice the Gumbel distribution should be used with caution for $P_f < 10^{-6}$.

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